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A CLASSIFICATION OF NATURAL VECTOR BUNDLE MORPHISMS $F \circ \otimes_s^q \to \otimes_s^q \circ F$ OVER id_F

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Abstract

In this paper, we give a classification of natural vector bundle morphisms $F \circ \otimes_s^q \to \otimes_s^q \circ F$ over id_F, for an arbitrary product preserving gauge bundle functor F on vector bundles.

1. Introduction

In [5], we determine all natural vector bundle morphisms $T_A \circ \otimes_s^q \to \otimes_s^q \circ T_A$ over id_{T_A} associated to a Weil functor T_A and present applications to prolongation of tensor fields. By a natural vector bundle morphism $\tau : T_A \circ \otimes_s^q \to \otimes_s^q \circ T_A$ over id_{T_A}, we mean a system of base-preserving vector bundle morphisms, $\tau_E : T_A(\otimes_s^q E) \to \otimes_s^q (T_A E)$, for

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vector bundles E with standard fiber $\mathbb{R}^n$, such that $\otimes_s^q (T_A f) \circ \tau_E = \tau_F \circ T_A(\otimes_s^q f)$, for vector bundle morphisms f between such vector bundles.

The main fact used in the proof of Proposition 3.1 [5] (classification of natural vector bundle morphisms $T_A \circ \otimes_s^q \to \otimes_s^q \circ T_A$ over id_{T_A}) is that each Weil functor T_A induces a product preserving gauge bundle functor

$T_A : \mathcal{VB} \to \mathcal{FM}$ defined by

$$\begin{cases} T_A(E, M, \pi) = (T_A E, M, p_E), \\ \text{and} \\ T_A(\bar{f}, f) = (\bar{f}, T_A f), \end{cases} \tag{1.1}$$

where $p_E = \pi \circ \pi_{A,E} = \pi_{A,M} \circ T_A(\pi)$.

Replacing T_A by an arbitrary product preserving gauge bundle functor $F : \mathcal{VB} \to \mathcal{FM}$, one can see that Proposition 3.1, [5] is a particular case of a more general result: "natural vector bundle morphisms $F \circ \otimes_s^q \to \otimes_s^q \circ F$ over id_F are in bijection with equivariant linear maps $F(\otimes_s^q V) \to \otimes_s^q(V)$ (V real vector space of finite dimension)".

2. Product Preserving Gauge Bundle Functor on Vector Bundles

2.1. Weil algebra

A Weil algebra is a finite-dimensional quotient of the algebra of germs $\mathcal{E}_p = C_0^\infty(\mathbb{R}^p, \mathbb{R})(p \in \mathbb{N})$.

We denote by $\mathcal{M}_p$ the ideal of germs vanishing at 0; $\mathcal{M}_p$ is the maximal ideal of the local algebra $\mathcal{E}_p$.

Equivalently, a Weil algebra is a real commutative unital algebra such that $A = \mathbb{R} \cdot 1_A \bigoplus N$, where N is a finite dimensional ideal of nilpotent elements.

Example 2.1. (1) $\mathbb{R}$ is a Weil algebra since it is canonically isomorphic to the quotient $\mathcal{E}_p / \mathcal{M}_p$.

(2) $J_0^r(\mathbb{R}^p, \mathbb{R}) = \mathcal{E}_p / \mathcal{M}_p^{r+1}$ is a Weil algebra.

2.2. Product preserving gauge bundle functor on $\mathcal{VB}$

Let $F : \mathcal{VB} \to \mathcal{FM}$ be a covariant functor from the category $\mathcal{VB}$ of all vector bundles and their vector bundle homomorphisms into the category $\mathcal{FM}$ of fibered manifolds and their fibered maps. Let $B_{\mathcal{VB}} : \mathcal{VB} \to \mathcal{M}f$ and $B_{\mathcal{FM}} : \mathcal{FM} \to \mathcal{M}f$ be the respective base functors.

A *gauge bundle functor on* $\mathcal{VB}$ is a functor F satisfying $B_{\mathcal{FM}} \circ F = B_{\mathcal{VB}}$ and the localization condition: For any vector bundle (E, M, π) and any inclusion of an open vector subbundle $i : \pi^{-1}(U) \hookrightarrow E$, $F\pi^{-1}(U)$ is isomorphic to $p_E^{-1}(U)$ over U and the map Fi can be identified to the inclusion $p_E^{-1}(U) \to FE$.

Given two gauge bundle functors F_1, F_2 on $\mathcal{VB}$, by *a natural transformation* $\tau : F_1 \to F_2$, we shall mean a system of base preserving fibered maps $\tau_E : F_1E \to F_2E$ for every vector bundle E satisfying $F_2 f \circ \tau_E = \tau_G \circ F_1 f$ for every vector bundle morphism $f : E \to G$.

A gauge bundle functor F on $\mathcal{VB}$ is *product preserving*, if for any product projections $E_1 \xleftarrow{pr_1} E_1 \times E_2 \xrightarrow{pr_2} E_2$ in the category $\mathcal{VB}$, $FE_1 \xleftarrow{Fpr_1} F(E_1 \times E_2) \xrightarrow{Fpr_2} FE_2$ are product projections in the category $\mathcal{FM}$. In the other words, the map $(Fpr_1, Fpr_2 : F(E_1 \times E_2) = F(E_1) \times F(E_2))$ is a fibered isomorphism over $id_{M_1 \times M_2}$.

Example 2.2. (a) Each Weil functor T_A induces a product preserving gauge bundle functor $T_A : \mathcal{VB} \to \mathcal{FM}$ by (1.1).

(b) Let $A = \mathbb{R} \cdot 1_A \bigoplus N$ be a Weil algebra and V be an A-module such that $\dim_{\mathbb{R}}(V) < \infty$. For a vector bundle (E, M, π) and $x \in M$, let

$$T_x^{A,V}E = \{(\varphi_x, \psi_x)/\varphi_x \in Hom(C_x^{\infty}(M, \mathbb{R}), A),$$

and

$$\psi_x \in Hom_{\varphi_x}(C_x^{\infty, f.l}(E), V)\},$$

where $Hom(C_x^{\infty}(M, \mathbb{R}), A)$ is the set of algebra homomorphisms φ_x from the algebra $C_x^{\infty}(M, \mathbb{R}) = \{germ_x(g)/g \in C^{\infty}(M, \mathbb{R})\}$ into A and $Hom_{\varphi_x}(C_x^{\infty, f.l}(E), V)$ is the set of module homomorphisms ψ_x over φ_x from the $C_x^{\infty}(M, \mathbb{R})$-module $C_x^{\infty, f.l}(E, \mathbb{R}) = \{germ_x(h)/h : E \to \mathbb{R}$ is fiber linear$\}$ into $\mathbb{R}$. Let $T^{A,V}E = \bigcup_{x \in M} T_x^{A,V}E$ and $p_E^{A,V} : T^{A,V}E \to M$, $(\varphi, \psi) \ni T_x^{A,V}E \mapsto x$. $\left(T^{A,V}E, M, p_E^{A,V}\right)$ is a well-defined fibered manifold. Indeed, let $c = (\pi^{-1}(U), x^i \circ \pi, y^j)$, $1 \le i \le m$, $1 \le j \le n$ be a fibered chart of E; then the map

$$\phi_c : (p_E^{A,V})^{-1}(U) \to U \times N^m \times V^n$$

$$(\varphi_x, \psi_x) \mapsto (x, \varphi_x(germ_x(x^i - x^i(x))), \psi_x(germ_x(y^j)));$$

is a local trivialization of $T^{A,V}E$. Given another vector bundle (G, N, π') and a vector bundle homomorphism $f : E \to G$ over $\bar{f} : M \to N$, let

$$T^{A,V}f : T^{A,V}E \to T^{A,V}G$$

$$(\varphi_x, \psi_x) \mapsto (\varphi_x \circ \bar{f}_x^*, \psi_x \circ f_x^*),$$

where $\bar{f}_x^* : C_{\bar{f}(x)}^{\infty}(N) \to C_x^{\infty}(M)$ and $f_x^* : C_{\bar{f}(x)}^{\infty, f.l.}(G) \to C_x^{\infty, f.l.}(E)$ are given by the pull-back with respect to $\bar{f}$ and f. Then $T^{A,V}f$ is a fibered map over $\bar{f}$. $T^{A,V} : \mathcal{VB} \to \mathcal{FM}$ is a product preserving gauge bundle functor (see [3]).

Remark 2.1. (a) Each product preserving gauge bundle functor $F : \mathcal{VB} \to \mathcal{FM}$ associates a pair $\left(A^F, V^F\right)$, where $A^F = F(id_{\mathbb{R}} : \mathbb{R} \to \mathbb{R})$ is a Weil algebra and $V^F = F(\mathbb{R} \to pt)$ is an A^F-module such that $\dim_{\mathbb{R}}\left(V^F\right) < \infty$. Moreover, there is a natural isomorphism $\ominus : F \to T^{A^F, V^F}$ and equivalence classes of functors F are in bijection with equivalence classes of pairs $\left(A^F, V^F\right)$. In particular, the product preserving gauge bundle functor (1.1) is equivalent to $T^{A,A}$.

(b) Each product preserving gauge bundle functor $F : \mathcal{VB} \to \mathcal{FM}$ induces a Weil functor $G^F : \mathcal{M}f \to \mathcal{FM}$ (equivalent to T_{A^F}) given by

$$\begin{cases} G^F(M) = F(M, M, id_M) = (FM, M, \pi_M), \\ \text{and} \\ G^F(f) = (f, f) = (f, Ff). \end{cases}$$

3. Natural Vector Bundle Morphisms $F \circ \otimes_s^q \to \otimes_s^q \circ F$ Over id_F

In this section, $F : \mathcal{VB} \to \mathcal{FM}$ is a product preserving gauge bundle functor.

3.1. Preliminaries

We write for $\mathcal{VB}$ the subcategory of $\mathcal{FM}$ of vector bundles and vector bundle morphisms; $\mathcal{D}$ is the subcategory of $\mathcal{VB}$ of vector bundles with the standard fiber V and morphisms of vector bundles, which are isomorphisms on fibers.

Let us consider the following vector spaces:

$$F(\otimes_s^q V) := F\left((\overset{s}{\otimes} V^*) \otimes (\overset{q}{\otimes} V)\right); \quad \otimes_s^q (F(V)) := \left(\overset{s}{\otimes}(FV)^*\right) \otimes \left(\overset{q}{\otimes} FV\right),$$

where V is the vector bundle $V \to pt$ (pt one-point manifold).

If φ is a linear automorphism of V (i.e., a vector bundle over id_{pt}), one can consider the following linear automorphisms:

$$F(\otimes_s^q \varphi) := F\left(\overset{s}{\otimes}({}^t\varphi^{-1}) \otimes (\overset{q}{\otimes} \varphi)\right) \text{ and } \otimes_s^q (F\varphi) := \overset{s}{\otimes}({}^t(F\varphi)^{-1}) \otimes \left(\overset{q}{\otimes} F\varphi\right),$$

respectively, on $F(\otimes_s^q V)$ and $\otimes_s^q (F(V))$.

Finally, let us consider the functors $F \circ \otimes_s^q : \mathcal{D} \to \mathcal{VB}$ and $\otimes_s^q \circ F : \mathcal{D} \to \mathcal{VB}$ defined as follows:

$$\begin{cases} F \circ \otimes_s^q((E, M, \pi)) = (F(\otimes_s^q E), FM, F(\otimes_s^q \pi)), \\ F \circ \otimes_s^q((\bar{f}, f)) = (F\bar{f}, F(\otimes_s^q f)), \end{cases}$$

and

$$\begin{cases} \otimes_s^q \circ F((E, M, \pi)) = (\otimes_s^q (FE), FM, \otimes_s^q (F\pi)), \\ \otimes_s^q \circ F((\bar{f}, f)) = (F\bar{f}, \otimes_s^q (Ff)). \end{cases}$$

Note that $\mathcal{D}$ may be replaced with $\mathcal{VB}$ in the case $s = 0$.

3.2. Natural vector bundle morphisms $F \circ \otimes_s^q \to \otimes_s^q \circ F$ over id_F

Let us consider the representation $\rho_{q,s,V} : GL(V) \to GL(\otimes_s^q V)$ given by $\rho_{q,s,V}(u) = \otimes_s^q(u)$. Let us denote $\lambda_V : GL(V) \times V \to V, (u, x) \mapsto u(x)$ the canonical linear action (this is a vector bundle morphism over $GL(V) \times pt \to pt$, where pt denotes an one-point manifold); the map $F\lambda_V : FGL(V) \times FV \to FV$ is also a linear action, so there is a unique representation, $j_V : FGL(V) \to GL(FV)$, defined by $j_V(\tilde{g})(\tilde{v}) = F\lambda_V(\tilde{g}, \tilde{v})$. The representation

$$j_{\otimes_s^q V} \circ F\rho_{q,s,V} : FGL(V) \to GL(F(\otimes_s^q V)),$$

will be denoted $(\rho_{q,s,V})_1$; $(\rho_{q,s,V})_1$ induces a left action of $FGL(V)$ on $F(\otimes_s^q V)$ defined by $\tilde{g} \cdot T_1 = (\rho_{q,s,V})_1(\tilde{g})(T_1)$. The representations $\rho_{q,s,FV}$ and j_V also induce a left action of $FGL(V)$ on $\otimes_s^q FV$ defined by $\tilde{g} \cdot \tilde{T} = \rho_{q,s,FV}(j_V(\tilde{g}))(\tilde{T})$.

Definition 3.1. A linear map $\overline{\tau} : F(\otimes_s^q V) \to \otimes_s^q(FV)$ is said to be *equivariant*, if it is $FGL(V)$-equivariant with respect to the previous actions, i.e.,

$$\rho_{q,s,FV}(j_V(\tilde{g})) \circ \overline{\tau} = \overline{\tau} \circ (\rho_{q,s,V})_1(\tilde{g}), \tag{3.1}$$

for all $\tilde{g} \in FGL(V) \cong T_{A^F} GL(V)$.

Definition 3.2. A natural vector bundle morphism $\tau : F \circ \otimes_s^q \to \otimes_s^q \circ F$ over id_F is a system of base-preserving vector bundle morphisms, $\tau_E : F(\otimes_s^q E) \to \otimes_s^q(FE)$, for every $\mathcal{D}$-object, such that $\otimes_s^q(Ff) \circ \tau_E = \tau_G \circ F(\otimes_s^q f)$, for each $\mathcal{D}$-morphism $f : E \to G$.

Proposition 3.1. *There is a bijective correspondence between the set of all natural vector bundle morphisms* $\tau : F \circ \otimes_s^q \to \otimes_s^q \circ F$ *over* id_F *and the set of all equivariant linear maps* $F(\otimes_s^q V) \to \otimes_s^q(FV)$.

Proof. Let $\overline{\tau} : F(\otimes_s^q V) \to \otimes_s^q(FV)$ be an equivariant linear map and $\varphi : \pi^{-1}(U) \to U \times V$ be a local trivialisation of a vector bundle (E, M, π); let

$$\tau_E \,|\, [F(\otimes_s^q \pi)]^{-1}(FU) = (\otimes_s^q F\varphi^{-1}) \circ (id_{FU} \times \overline{\tau}) \circ F(\otimes_s^q \varphi). \tag{3.2}$$

(1°) <u>The right hand of (3.2) does not depend on φ:</u> Indeed, let $\varphi_1 : \pi^{-1}(U) \to U \times V$ be another local trivialisation such that $(\varphi_1 \circ \varphi^{-1})(x, t) = (x, a(x) \cdot t)$; one has

$$\begin{cases} (\otimes_s^q \varphi_1) \circ (\otimes_s^q \varphi^{-1})(x, T) = (x, \rho_{q,s,V}(a(x)) \cdot T), \\ (F\varphi_1 \circ F\varphi^{-1})(\tilde{x}, \tilde{t}) = (\tilde{x}, j_V(Fa(\tilde{x})) \cdot \tilde{t}), \\ F(\otimes_s^q \varphi_1) \circ F(\otimes_s^q \varphi^{-1})(\tilde{x}, \tilde{T}) = (\tilde{x}, (\rho_{q,s,V})_1(Fa(\tilde{x})) \cdot \tilde{T}), \\ \otimes_s^q (F\varphi_1) \circ \otimes_s^q (F\varphi^{-1})(\tilde{x}, T_1) = (\tilde{x}, \rho_{q,s,FV}(j_V(Fa(\tilde{x}))) \cdot T_1). \end{cases}$$

(2°) <u>τ is a natural vector bundle morphism</u>: Indeed, let $f : E \to E'$ be a $\mathcal{D}$-morphism over $\bar{f}$, $\varphi : \pi^{-1}(U) \to U \times V$ be a local trivialisation of E, and $\varphi' : (\pi')^{-1}(U') \to U' \times V$ be a local trivialisation of E' such that $\bar{f}(U) \subset U'$. Let us put $(\varphi' \circ f \circ \varphi^{-1})(x, t) = (\bar{f}(x), b(x) \cdot t)$. For any $(\tilde{x}, \tilde{T}) \in F(\otimes_s^q \varphi) \circ (F(\otimes_s^q \pi))^{-1}(FU)$,

$$\begin{aligned} &(\otimes_s^q (F\varphi')) \circ \otimes_s^q (Ff) \circ \tau_E \circ F(\otimes_s^q \varphi^{-1})(\tilde{x}, \tilde{T}) \\ &\qquad = (F\bar{f}(\tilde{x}), \rho_{q,s,FV}(j_V(Fb(\tilde{x}))) \cdot \bar{\tau}(\tilde{T})), \end{aligned}$$

and

$$\begin{aligned} &(\otimes_s^q F\varphi') \circ \tau_{E'} \circ F(\otimes_s^q f) \circ F(\otimes_s^q \varphi^{-1})(\tilde{x}, \tilde{T}) \\ &\qquad = (F\bar{f}(\tilde{x}), \bar{\tau} \circ (\rho_{q,s,V})_1(Fb(\tilde{x})) \cdot \tilde{T}); \end{aligned}$$

but $\bar{\tau}$ is equivariant, hence $\otimes_s^q (Ff) \circ \tau_E = \tau_{E'} \circ F(\otimes_s^q f)$. Furthermore, $\tau_{V \to pt} = \bar{\tau}$.

(3°) The map $\Psi : \bar{\tau} \mapsto \tau$, is obviously injective. The surjection can be shown as follows. Indeed, given a natural vector bundle morphism $\tau : F \circ \otimes_s^q \to \otimes_s^q \circ F$ over id_F, let us define the map $\bar{\tau} : F(\otimes_s^q V) \to \otimes_s^q (FV)$ by $\bar{\tau} = \tau_{V \to pt}$.

(i) For a linear automorphism φ of V, we have $\otimes_s^q(F\varphi)\circ\overline{\tau} = \overline{\tau}\circ F(\otimes_s^q\varphi)$: Indeed, φ is a $\mathcal{D}$-morphism over id_{pt}, so $\otimes_s^q(F\varphi)\circ\tau_{V\to pt} = \tau_{V\to pt}\circ F(\otimes_s^q\varphi)$.

(ii) $\underline{\tau_{\mathbb{R}^m\times V\to\mathbb{R}^m} = id_{F\mathbb{R}^m}\times\overline{\tau}}$: Indeed, the projection $pr_2:\mathbb{R}^m\times V\to V$, $(x, t)\mapsto t$, is a $\mathcal{D}$-morphism (over $\mathbb{R}^m\to pt$), hence

$$F(pr_2) = pr_2 : F\mathbb{R}^m\times FV\to FV,$$

is also a $\mathcal{D}$-morphism. Moreover,

$$\otimes_s^q(pr_2) = pr_2 : \mathbb{R}^m\times(\otimes_s^q V)\to(\otimes_s^q V),$$

then

$$F(\otimes_s^q(pr_2)) = pr_2 : F\mathbb{R}^m\times F(\otimes_s^q V)\to F(\otimes_s^q V).$$

The relation $\otimes_s^q(F(pr_2))\circ\tau_{\mathbb{R}^m\times V\to\mathbb{R}^m} = \tau_{V\to pt}\circ F(\otimes_s^q(pr_2))$ can be written $pr_2\circ\tau_{\mathbb{R}^m\times V\to\mathbb{R}^m} = \overline{\tau}\circ pr_2$, hence $\tau_{\mathbb{R}^m\times V\to\mathbb{R}^m} = id_{F\mathbb{R}^m}\times\overline{\tau}$.

(iii) $\underline{\overline{\tau}\text{ is equivariant}}$: Let $A^F = \mathcal{E}_p/I$ be the Weil algebra associated to F. Taking $f(x, t) = (x, a(x)\cdot t)$, where $a:\mathbb{R}^p\to GL(V)$ is C^∞, the relation

$$\otimes_s^q(Ff)\circ\tau_{\mathbb{R}^p\times V\to\mathbb{R}^p} = \tau_{\mathbb{R}^p\times V\to\mathbb{R}^p}\circ F(\otimes_s^q f),$$

is equivalent to

$$\rho_{q,s,FV}(j_V(Fa(\widetilde{x})))\circ\overline{\tau} = \overline{\tau}\circ\left(\rho_{q,s,V}\right)_1(Fa(\widetilde{x})),$$

for $\widetilde{x}\in F\mathbb{R}^p\cong T_{A^F}\mathbb{R}^p$. But for $\widetilde{g}_1\in T_{A^F}GL(V)$, there is $a\in C^\infty(\mathbb{R}^p, GL(V))$ and $\widetilde{x}_1\in T_{A^F}\mathbb{R}^p$ such that $\widetilde{g}_1 = T_{A^F}a(\widetilde{x})$ (see [5]), hence (3.1) is clear by using the natural equivalence $G^F\to T_{A^F}$.

(iv) $\underline{\Psi(\overline{\tau}) = \tau}$: Indeed, each local trivialisation $\varphi : \pi^{-1}(U) \to U \times V$ of a vector bundle (E, M, π) is a $\mathcal{D}$-morphism over id_U, hence

$$\otimes_s^q(F\varphi) \circ \tau_{E|_U} = \tau_{U \times V \to U} \circ F(\otimes_s^q \varphi) = (id_{FU} \times \overline{\tau}) \circ F(\otimes_s^q \varphi), \quad \text{by} \quad \text{(ii)}$$

i.e., $\tau_{E|_U} = \Psi(\overline{\tau})_{E|_U}$, according to (3.2). □

4. Equivariant Linear Maps $F(\otimes_s^q V) \to \otimes_s^q(FV)$

Let $F : \mathcal{VB} \to \mathcal{FM}$ is a product preserving gauge bundle functor.

4.1. The case $q = s = 0$

An equivariant linear map $\overline{\tau} : F(\otimes_0^0 V) \to \otimes_0^0(FV)$ is simply a linear form $\overline{\tau} : A^F = F\mathbb{R} \to \mathbb{R}$ since $\left(\rho_{0,0,V}\right)_1(\widetilde{g}) = id_{A^F}$ and $\rho_{0,0,FV}(j_V(\widetilde{g})) = id_{\mathbb{R}}$. Moreover, each linear form $i \in \left(A^F\right)^*$ defines an equivariant linear map $\overline{\tau} : F(\otimes_0^0 V) \to \otimes_0^0(FV)$ by $\overline{\tau} = i$.

4.2. The case $q = 1, s = 0$

A linear map $\overline{\tau} : FV \to FV$ is equivariant if and only if

$$j_V(\widetilde{g}) \circ \overline{\tau} = \overline{\tau} \circ j_V(\widetilde{g}), \tag{4.1}$$

for any $\widetilde{g} \in FGL(V)$. For a fixed element $a \in A^F$, one can define an equivariant linear map $\overline{\tau}_a : FV \to FV$ by

$$\overline{\tau}_a(v) = a \cdot v = F(m)(a, v),$$

where the multiplication map $m : \mathbb{R} \times V \to V$ is viewed as a vector bundle morphism over $\mathbb{R} \times pt \to pt$; moreover $\overline{\tau}_a \circ \overline{\tau}_b = \overline{\tau}_b \circ \overline{\tau}_a = \overline{\tau}_{ab}$, hence $\overline{\tau}_a$ is an A^F-module endomorphism.

Proposition 4.1. *Equivariant linear maps* $F\mathbb{R}^n \to F\mathbb{R}^n$ *are in bijection with* A^F*-module endomorphisms* $\nu : V^F \to V^F$ *such that* ν^n *satisfies* (4.1).

Proof. A natural vector bundle morphisms $\tau : F \to F$ over id_F is a natural transformations $F \to F$ over id_F between product preserving gauge bundle functors on $\mathcal{VB}$, hence there is a morphism of pairs $(\mu, \nu) : (A^F, V^F) \to (A^F, V^F)$ with $\mu = id_{A^F}$ such that $\tau = \Theta^{-1} \circ \tau^{\mu, \nu} \circ \Theta$ ([3], Theorem 2). Since

$$\overline{\tau} = \tau_{\mathbb{R}^n \to pt} = (\tau_{\mathbb{R} \to pt})^n = \nu^n,$$

the result follows. □

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